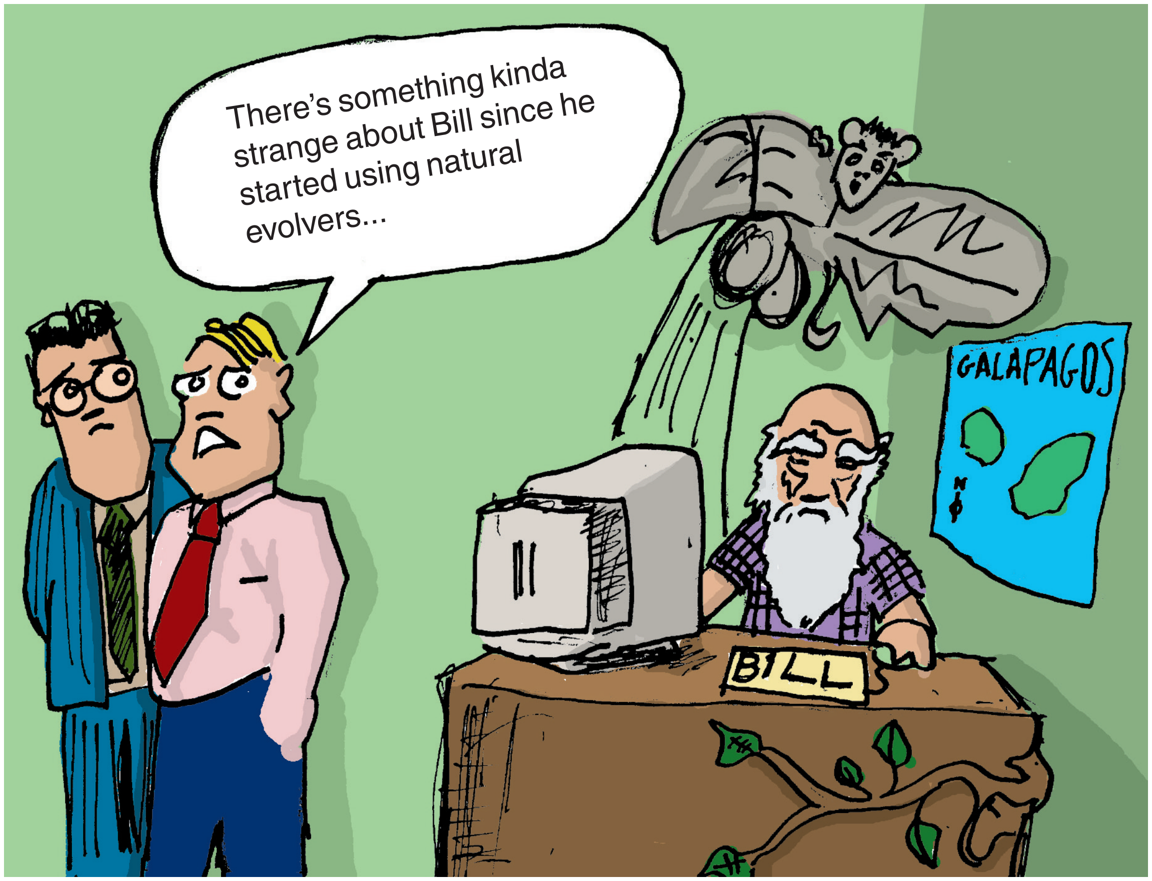
Aaron Brown

The Dakota Option Part II

The all-American exploration of Tikhonov's methods continues

Last issue I introduced the concept of a Dakota option, a cash settlement option with a fixed payout at expiration, in which the holder has the right to exercise at any time before or *after* expiration at the Black-Scholes price. Pricing and hedging Dakotas requires solving the backward heat equation, a classic ill-posed problem. This month we'll investigate a technique appropriate for one Dakota example. Next issue I will complete the series by using another technique to solve a different Dakota example.



Why?

What is the point of this exercise? Do I really think Dakota settlement is a useful trading idea? Yes and no. Yes, I think with some further work, the Dakota idea can be used to reduce settlement disputes in options on illiquid assets and to help price derivatives with trading restrictions. Perhaps it will even be more useful than that, if someone picks up the idea and advances it. It might be particularly interesting with respect to real options. But no, that's not why I wrote this article.

What I do want to do is proselytize for Andrei Tikhonov's methods of applied mathematics, which deserve to be known better. If you find the mathematical approaches in the article interesting, read some Tikhonov to learn the real thing from a great genius.

Another rationale for this article is to demonstrate the importance of give and take between abstract quantitative reasoning and practical financial reality. A good quant must understand both. Applied mathematics is *more* than mathematics, not less.

And finally, I am not writing this article to help employee option recipients find loopholes in their corporate trading policies. I see no reason to do that; I support stronger corporate trading policies myself. Moreover, the example treated here would be unlikely to impress any compliance officer. Don't call me from jail if you try it and fail. It's a textbook example, nothing more.

Bill's Megacomp options

The first example we considered last month is Bill's desire to write a 20-year derivative on Megacomp stock such that the value in 10 years is Max$(S-100, 0)$, where S is Megacomp's stock price. He plans to use this to hedge his 10-year employee stock options. He is not allowed to write contracts that depend on the price of Megacomp stock during his employment. I argued that there is no economically reasonable exact solution to this problem, but we can give Bill some help by using natural evolvers.

Consider any derivative security, that is any security whose price can be expressed at some point in time as a function of an underlying. Call the time at which the derivative price is defined "expiry." In general, it is easy to compute the value of the derivative at any point earlier than expiry. In theory it is possible to compute the value of the derivative after expiry if and only if the price function is analytic (that is, that it has continuous derivatives of all orders). Call options, for example, do not meet this condition because at expiry the first derivative is discontinuous at the exercise price.

However, even for analytic functions, there is no guarantee that you can compute the price into the indefinite future. For example, the Black-Scholes value of a call option, computed at a time before expiry, is analytic. But as you let the price evolve forward in time, the graph gets kinkier, until you reach the discontinuous first derivative at expiry. Many other analytic functions can evolve only for limited times into the future. Other functions can be evolved in theory, but become unstable or hard to compute. Still other functions lead to economically unreasonable prices in the real world.

Natural evolution

Functions whose prices evolve smoothly forever, in analytically tractable, numerically stable, economically valid ways are called natural evolvers. One common way to solve ill-posed problems is to try to approximate them with solutions of well-posed problems. In this case it means that instead of looking for a payoff function in 20 years that gives the exact correct values in 10 years, approximate the 10-year value function with natural evolvers. We know we'll have no trouble, either theoretically or practically, in computing the value of the approximation 20 years from now.

In the Black-Scholes world, the order zero natural evolver is $f(S) = 1$, the constant function. The present value of receiving \$1 at future time t is e^{-rt}, where r is the interest rate. The first order natural evolver is $f(S) = S$. The present value of receiving S_t at time t is $S_0 e^{-pt}$, where p is the payout rate on the underlying (the dividend yield on a stock, for example) and S_0 is the price today.

The second order natural evolver is $f(S) = \ln(S)$, the natural logarithm function. BS assumes that S follows geometric Brownian motion with mean r (the risk free rate of interest) and variance σ^2. That makes $\ln(S)$ a Normal random walk with mean $r - \sigma^2/2$ and variance σ^2. The present value of receiving $\ln(S_t)$ at time t is $e^{-rt}[\ln(S_0) + (r - \sigma^2/2)t]$. This value can be negative because "receiving" $\ln(S)$ can mean paying money if $S < 1$.

The first two functions, $f(S) = 1$ and $f(S) = S$, are clearly reasonable economically for liquid securities. It's true that interest rates aren't constant, and payout rates aren't constant, known or continuous. But these deviations from the BS assumptions are minor in most cases of interest. Moreover, there is a wide range of instruments available to allow a good trader to lock in interest and payout rates for the purposes of specific transactions.

Instead of looking for a payoff function in 20 years that gives the exact correct values in 10 years, approximate the 10-year value function with natural evolvers

Think like a trader

To test the economics of the third function, $f(S) = \ln(S)$, we have to think like a trader. If we ask a trader to deliver $\ln(S_t)$ at time t, she is going to buy \$1 worth of the underlying and keep the portfolio rebalanced to be worth \$1 at all times. Suppose the prices at which she rebalances are $S_0, S_1, S_2, \ldots, S_{n-1}, S_n (S_n = S_t)$. The prices need not be at equal intervals, in fact the rebalancing decision is more likely to depend on price movement (such as, rebalance every time the price moves more than 5 per cent) than time.

Suppose she puts all the rebalancing proceeds when the price goes up into an account, and withdraws from that account to pay rebalancing costs when the price goes down. By time t, the account will hold:

$$\sum_{i=1}^{n} \frac{S_i}{S_{i-1}} - 1$$

$\frac{S_i}{S_{i-1}}$ can be expanded in a Taylor series to:

$$1 + \ln\left(\frac{S_i}{S_{i-1}}\right) + \frac{1}{2}\ln^2\left(\frac{S_i}{S_{i-1}}\right) + \frac{1}{6}\ln^3\left(\frac{S_i}{S_{i-1}}\right) + O\left(\ln^4\left(\frac{S_i}{S_{i-1}}\right)\right)$$

so the account will hold approximately:

$$\sum_{i=1}^{n} \ln\left(\frac{S_i}{S_{i-1}}\right) + \frac{1}{2}\sum_{i=1}^{n} \ln^2\left(\frac{S_i}{S_{i-1}}\right)$$

The first term is $\ln\left(\frac{S_n}{S_0}\right)$ and the second is $\frac{\sigma^2 t}{2}$ where $\sigma\sqrt{t}$ is the actual volatility of S over the period. The error will be negligible as long as the price gaps between rebalancing are not large. For example, even a 50 per cent change between p_i and p_{i+1} results in an error term about 2.5 per cent the size of the two terms above. For professional traders dealing in liquid securities, it is generally possible to rebalance at far smaller price gaps.

>

There is one more factor to consider, interest. It costs \$1 today to set up the portfolio, that has a present value of \$1. The portfolio is still worth \$1 at time t, that has a present value of e^{-rt}. The other amounts are earned over the interval from now until time t, so their present value is $\frac{1-e^{-rt}}{rt}$ times their nominal values. So to avoid arbitrage:

$$-1 + e^{-rt} + \frac{1-e^{-rt}}{rt}\left[\ln(S_t) - \ln(S_0) + \frac{\sigma^2 t}{2}\right]$$

must be worth zero which implies $\ln(S_t)$, paid at time t, is worth:

$$\ln(S_0) + \left(r - \frac{\sigma^2}{2}\right)t$$

The present value of this amount is:

$$e^{-rt}\left[\ln(S_0) + \left(r - \frac{\sigma^2}{2}\right)t\right]$$

This is the same price derived from theory.

The trader takes some risk in quoting us this price. Interest rates and volatility may not be constant, and if the price jumps she may accumulate significant rebalancing error. Moreover, she has all the transaction costs from the hedge. However, these factors are common to all derivative trading, and lighter for this security than for most. The biggest risks, vega and jumps, can be hedged. So the theoretical price for this security should be a good predictor of its market price.

On additional pricing issue is that S can go to zero, in which case $\ln(S)$ goes to negative infinity. How can we price a capped security, with a maximum negative payout? If S is a stock, we can do a quick-and-dirty adjustment by using the market rate for the firm's debt instead of the risk-free rate for the r inside the brackets in the formula. This is not exact, because default on a debt instrument is not the same thing as the stock price falling below some low value. But it is more accurate than a theoretical calculation assuming the stock price continues following a log random walk with constant parameters at very low values. And it gives a quick ballpark estimate of the size of the cap effect on pricing.

POSITION AT THE END OF 10 YEARS

Stock Price	Option Value	Stock Position	Cash Position	Log Position	Net Position
10	0	−1,100,000	−9,786,673	9,397,716	−1,488,957
20	0	−2,200,000	−9,786,673	12,371,351	384,678
25	0	−2,750,000	−9,786,673	13,328,647	791,975
50	0	−5,500,000	−9,786,673	16,302,282	1,015,610
75	0	−8,250,000	−9,786,673	18,041,747	5,074
100	0	−11,000,000	−9,786,673	19,275,917	−1,510,756
150	5,000,000	−16,500,000	−9,786,673	21,015,382	−271,291
200	10,000,000	−22,000,000	−9,786,673	22,249,552	462,879
500	40,000,000	−55,000,000	−9,786,673	26,180,483	1,393,811
1,000	90,000,000	−110,000,000	−9,786,673	29,154,118	−632,554

Bill's Problem

So how does all this help Bill? He has been given a 10-year call on 100,000 shares of Megacomp at an exercise price of \$100 per share, which is also the current price. Assuming 32 per cent annual volatility, no dividends and a 4 per cent risk-free interest rate, these options are worth \$5.1 million today. But the holding is risky. Using the risk-neutral assumptions Bill has a 46 per cent chance of getting nothing at all from the options, and a 78 per cent chance of getting less than \$1 million. His standard deviation is over \$13 million. Using realistic actual rates of return will give a higher probability of exercise, but also a higher standard deviation of eventual value.

Bill wants to hedge this position using an option that depends only on the value of Megacomp stock in 20 years. Let's say he promises to deliver in 20 years 110,000 shares of Megacomp, plus \$14.6 million, but he wants to be paid \$6.4 million times the natural logarithm of Megacomp's stock price. He will

POSITION VALUE AFTER 1 YEAR AT STOCK PRICE = \$100

		Volatility				
		16%	24%	32%	40%	48%
Interest Rate	1%	4,461,965	3,974,080	2,757,934	784,411	−1,960,498
	2%	3,961,083	3,576,975	2,622,207	1,047,656	−1,167,474
	4%	3,146,804	2,820,003	2,193,269	1,175,868	−271,177
	8%	1,869,101	1,572,128	1,222,027	753,183	115,976
	16%	89,327	−2,663	−121,034	−254,446	−404,934

POSITION AT THE END OF 1 YEAR

Stock Price	Option Intrinsic Value	Option Time Value	Stock Position	Cash Position	Log Position	Net Position
2	0	4,561	−220,000	−6,827,930	1,437,710	−5,605,659
25	0	906,871	−2,750,000	−6,827,930	8,997,381	326,322
400	30,000,000	5,752,249	−44,000,000	−6,827,930	17,295,919	2,220,238
1,500	140,000,000	5,386,463	−165,000,000	−6,827,930	21,252,021	−5,189,446

receive for this promise:

$$\$11{,}000000 + \$14{,}600{,}000\, e^{-0.04\times 20} - \$6{,}400{,}000\, e^{-0.04\times 20} \times \left[\ln(100) + \left(.04 - \frac{0.32^2}{2}\right) \times 10\right]$$

or \$5.0 million. In 10 years the value of his position will depend on Megacomp's stock price:

We have not eliminated all of Bill's risk, there is a residual amount in 10 years that bounces around between plus and minus \$1.5 million. The standard deviation is \$0.9 million (\$0.7 million present value). But this is much better than the \$13 million standard deviation of the unhedged option position.

Greeks and the tails

The hedge does break down for extreme movements in Megacomp stock. The 0.1 per cent tail events are approximately a price above \$2,000 per share or below \$4 per share in 10 years. These will cost \$5 million or more, he will have to give back everything he got from entering into the position at inception. Because the events are so unlikely it would be cheap to cap Bill's losses, but that contract would have to depend on Megacomp's stock price during Bill's period of employment.

So Bill is going to have to keep an eye on the position to tame the tail risk. Look at some possible scenarios at the end of one year.

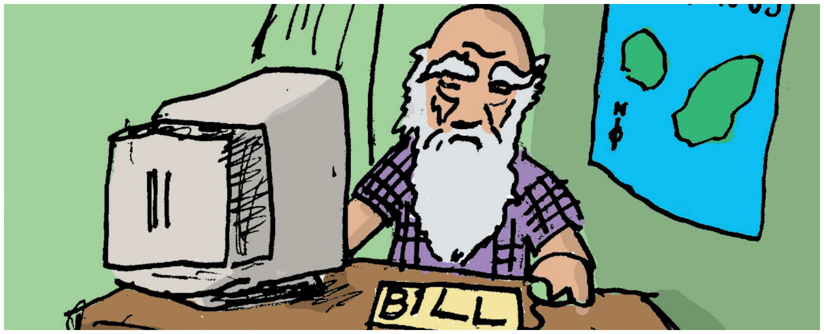
If Bill lets the Megacomp stock price get to \$2 or \$1,500 without acting, he's in trouble. It will cost him more than the \$5 million he received up front to buy his way out of his position. He'll have lost money and lost the options to boot (which only hurts in the \$1,500 case).

But if he acts when the stock price is \$25 or \$400, he can unwind his position at a profit. He can keep his \$5 million plus have residual value. The only catch is that the residual value is in the form of time value on his options. He'll have to write a check to get out of the hedge (\$580,000 if the price goes to \$25, \$3.5 million if the price goes to \$400), but it will be less than his initial \$5 million. At that point he could stop hedging the options, regarding them as likely worthless in the \$25 case and likely exercised in the \$400 case, or he could compute a hedge for his new position.

The next thing to consider is whether Bill's hedge will stand up to changes in parameters. After 1 year, here are the values of his total position (option plus hedge) at various interest rates and Megacomp implied volatilities. Remember, he has \$5 million in hand in addition to the numbers below.

The only worrisome case is falling interest rates combined with rising volatility. Once again, we have to rely on Bill to keep an eye on things and rebalance the hedge if parameters move too far away from the initial values.

What happens if Bill quits his job and has to exercise the options immediately or forfeit them? He'll have to take off the hedge at that point of course, and he'll lose any remaining time value of the options. In some cases, such as if he quits soon after receiving the options, the stock price has not moved much but volatility is up and interest rates are down, he may have to pay more



The only worrisome case is falling interest rates combined with rising volatility

than the \$5 million he received plus any profits from exercise to get out of the hedge. These are unlikely scenarios, but they can happen. So if Bill thinks he might quit or get fired, he should think hard before hedging his options.

Someone is going to object (I know, I see you in the back of the room with your hand waving and a smirk on your face) that if we're going to let Bill rebalance, we don't need such a complicated hedge. Bill could simply promise to deliver an amount of Megacomp stock in 20 years equal to the delta of his options, readjusting it for every price and parameter move. However, adding the log hedge reduces his hedging activity enormously. He probably will not have to rebalance at all, and if he does he can likely take off the hedge altogether.

Next Issue

In this article, we took a simple and unrealistic example. Bill wanted a Dakota option because he wanted the value defined 20 years from today for legal reasons, but was really concerned with the value 10 years from today. We solved it by changing the problem from evolving the price of a call option forward past expiry to looking for derivatives that evolve naturally, then approximating the call option with a sum of natural evolvers.

In the next issue we'll consider a more complicated example, but one closer to practical use: how to settle options when settlement at expiry is impossible due to market conditions or other problems. Natural evolvers will not help us here because we need an exact solution, not an approximation, and because even natural evolvers do not behave smoothly in illiquid markets. Instead we'll bound the problem and discretize it. The discretization is not an approximation or programming aid, but a fundamental change to the nature of the problem, one that makes it solvable.